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ARE FEM SOLUTIONS OF INCOMPRESSIBLE FLOWS REALLY INCOMPRESSIBLE? (OR HOW SIMPLE FLOWS CAN CAUSE HEADACHES!)

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SUMMARY

It is generally accepted that mixed and penalty finite element methods can routinely solve the incompressible Navier–Stokes equations. This paper shows by means of simple examples that problems can arise even for the simpler Stokes equations. The causes of the problem fall in either of two categories: round-off and ill conditioning, or a poor choice of pressure discretization. Nonsensical solutions can be obtained. Computation of the discrete divergence of the flow field is a simple and powerful tool to diagnose such conditions. In the first part of the paper several simple techniques for minimizing the effect of round-off are reviewed. In the second part it is shown that, for coupled flow problems, care must be exercised in the choice of the pressure approximation. A unified treatment of various observations by different workers is presented. This should prove useful for general users of the finite element method.

KEY WORDS Mixed and penalty FEM Navier–Stokes equations Round-off and ill conditioning Pressure discretization Coupled flow

INTRODUCTION

The work reported in this paper stems from preliminary attempts at solving the thermal–hydraulic equations governing the creeping flow of polymers. Such fluids are characterized by very high viscosities. Nonsensical solutions were obtained where mass was lost. In order to gain a better grasp of the problem, a simple flow solution was attempted: Poiseuille flow for a fluid of very large viscosity (see Figure 1 for a statement of the problem). A typical solution is shown in Figure 2. It is apparent that mass is lost progressively throughout the domain (from inlet to outlet). This led us to reassess the discrete divergence-free condition which was obviously violated in this case.

Moreover, other situations were observed in which the divergence-free condition was not satisfied, although the effects were not as spectacular as in the example of Figure 2. Gresho *et al.*[1] experienced difficulties with traditional elements for free convection problems. Very similar observations were described by Debbaut and Crochet[2] in flows of viscoelastic fluids and by Tidd *et al.*[3] for free surface problems.

Although the symptoms are identical in all cases (a poor satisfaction of the divergence-free condition), the underlying causes fall in two distinct categories: round-off and ill conditioning, or a poor choice of pressure discretization. In the Poiseuille flow round-off and ill conditioning are the culprit, whereas in the other cases the pressure is required to play a dual role: to enforce the continuity equation and to balance other forces (buoyancy effects for Boussinesq flows, for example).

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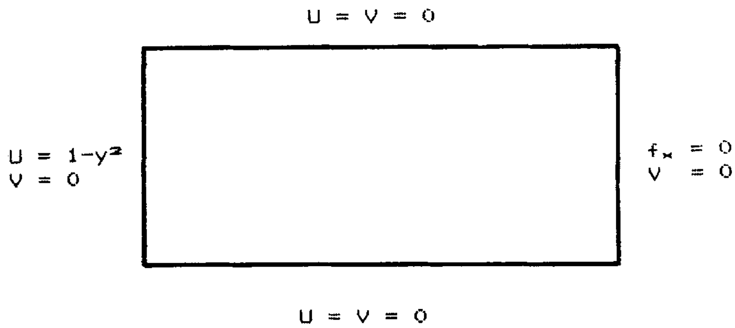


Figure 1. 2D Poiseuille flow

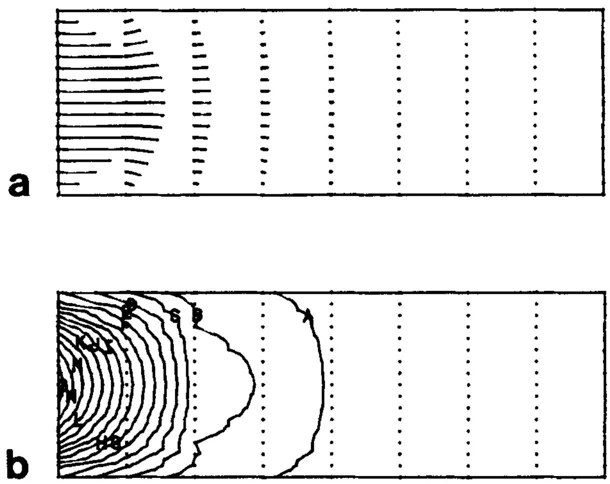


Figure 2. Penalty solution to Poiseuille flow. (a) velocity field; (b) pressure contours

This paper presents a unified treatment of many observations and comments made by various authors pertaining to problems with mass flow conservation. We show that, if care is not exercised, erroneous solutions can be obtained that could go undetected in difficult problems because of either complex geometry or intricate physics. This can have disastrous consequences in a design and analysis environment where the analyst is not necessarily a specialist of the finite element method. A general explanation is provided and a simple diagnostic tool to detect such situations which is easily incorporated in existing codes is presented. These should allow the user to recognize and avoid many of the common pitfalls of incompressible CFD. The comments presented in this paper should also help identify the better finite element approximations for the more complex coupled problems.

THE BASIC EQUATIONS

In order to provide a general treatment, we write the basic equations of motion in a slightly more general form than is commonly used. This will allow us to refer to the same set of equations throughout the paper, leaving only the description of the context to be made as is required. The momentum and continuity equations are written as

$$-2\nabla\cdot[\mu\mathbf{D}(\mathbf{U})]+\rho(\mathbf{U}\cdot\nabla)\mathbf{U}+\nabla P=\mathbf{S}(\mathbf{U}, T, \ldots), \tag{1}$$

$$\nabla\cdot\mathbf{U}=\mathbf{0}. \tag{2}$$

Boundary conditions complete the specification of the problem. $\mathbf{U}=(U_1, U_2, U_3)$ is the velocity vector, P the pressure, ρ the density, μ the viscosity, $\mathbf{S}$ a source term and $\mathbf{D}$ the rate of strain tensor defined by

$$D_{ij}(\mathbf{U})=\tfrac{1}{2}(U_{i,j}+U_{j,i})$$

The right-hand side of equation (1) is a problem-dependent source term. The simplest example is the isothermal flow of a Newtonian fluid with $\mathbf{S}$ being a body force.

A more complex situation occurs in free convection problems. In this case the source term is obtained from the Boussinesq approximation as

$$\mathbf{S}=\rho\beta\mathbf{g}(T-T_0). \tag{3}$$

The distribution of temperature T is obtained by supplementing equations (1) and (2) with the energy transport equation.

A last example is given by the flow of a viscoelastic fluid, for which

$$\mathbf{S}=-\nabla\cdot\boldsymbol{\tau}. \tag{4}$$

This expression describes the polymer contribution to the Cauchy stress tensor $\boldsymbol{\sigma}$

$$\boldsymbol{\sigma}=-P\mathbf{I}+2\mu\mathbf{D}+\boldsymbol{\tau}$$

($\boldsymbol{\tau}=\mathbf{0}$ for Newtonian fluids). A set of differential equations relating the components of $\boldsymbol{\tau}$ to the velocity field must be added to close the model. The upper convected Maxwell model is commonly used to define $\boldsymbol{\tau}$.[2] Other models exist that can be treated in a similar manner.

ROUND-OFF AND ILL CONDITIONING: THE STOKES PROBLEM

The Stokes equations are a prototypical model of creeping flow problems. They are obtained from the Navier–Stokes equations for small values of the Reynolds number $Re=\rho U_0 L/\mu$. It is generally accepted that these equations can be accurately and routinely solved by classical mixed or penalty Galerkin finite element methods. We will however show that vigilance is still important in this very simple case.

The symptoms

Consider the problem of planar Poiseuille flow with $\mu=10^5$. Such high values of the viscosity may occur locally in the case of variable property fluids and can have effects identical to those we present here for this somewhat contrived example.

The exact solution is $U(y)=1-y^2$, $V=0$ and a constant pressure gradient. The problem was solved using the finite element program CADYF.[4] The Q2/P1 element is used to solve the Navier–Stokes equations since it is one of the best elements for two-dimensional flows. The velocity is approximated by piecewise biquadratic polynomials, while the pressure is represented by a piecewise linear and discontinuous polynomial. A standard Galerkin formulation is used and the global system of equations is solved with either a Picard iteration or a Newton–Raphson method. Linearized systems are solved by Gaussian elimination.

Figure 3 presents the velocity and pressure solutions obtained with the mixed method, while Figure 2 gives those obtained with the penalty method using a value of 10^{-5} for the penalty parameter ε. It is clear from these results that both the penalty and the mixed method failed to produce an acceptable solution. Although the solution obtained with the mixed method seems acceptable, slight unphysical wiggles in the pressure contours are observed. A mass flow calculation shows that 1–2% mass is lost between the inflow and the outflow. In principle, mixed formulations guarantee global mass conservation. . . .

From the velocity vector plots one sees that the velocity field does not have zero divergence (i.e. the solution is not compatible with that of an incompressible fluid). Indeed, the discrete divergence of the solution was computed as follows for the linear pressure basis functions:

$$\max_E \left| \int_E \nabla \cdot \mathbf{U}_n \, d\Omega \right|,$$

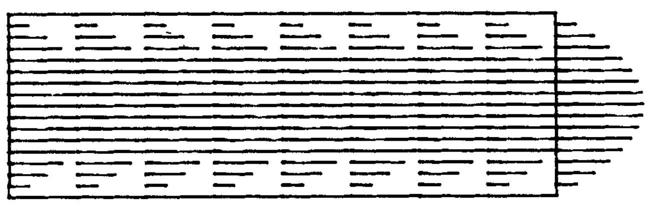
where $\mathbf{U}_n$ is the velocity vector on element E.

The maximum value of the components of the discrete divergence over all elements of the mesh was of the order of 10^{-3} for the mixed and the penalty methods, while theory predicts values of machine zero for mixed methods (10^{-14} for double precision on IBM computers) of order $O(\varepsilon)$ for penalty methods. Such values of the discrete divergence confirm a poor satisfaction of the incompressibility constraint.

Thus the systematic computation of a norm of the discrete divergence of the velocity field is a means of diagnosing a potentially disastrous consequence of poor satisfaction of mass conservation.

The above values of fluid properties cause an imbalance of the various terms in equation (1). Indeed, for creeping flows the controlling terms are the pressure gradient and the viscous stresses and, because the density is unity and the viscosity 10^5, the two terms are of wildly different magnitude.

a



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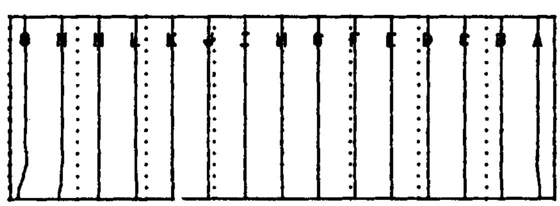

Figure 3. Mixed solution to Poiseuille flow: (a) velocity field; (b) pressure contours

Because computers operate with finite precision, these large differences will lead to important round-off error accumulation. The global matrix terms corresponding to the pressure terms will be very much smaller than those of the viscous terms. There is thus the possibility that, during the course of the matrix solution, the pressure terms will be lost because of floating point round-off errors.

Another interpretation consists in pointing out that round-off errors are small changes in the global matrix causing large changes in the solution. This is a well known symptom of ill conditioned systems of equations.[5] Hence ill conditioning plays a strong role. This can be especially serious when a penalty method is used, since it is well known that this method leads to poorly conditioned systems of equations.[6]

Some cures

Choosing an adequate dimensionless form. The use of a dimensionless form of the equations of motion is very advantageous because:

1. they provide an estimate of the order of difficulty of the problem being solved
2. they provide a measure of the relative importance of the terms in the equations, i.e. identify the important physical phenomena
3. they can reduce the potentially large differences in orders of magnitude that may occur between various terms in the field equations, i.e. a means of performing proper scaling of the terms.

Consider the following choice of reference quantities: velocity U_0, say the maximum velocity; length L_0, the width of the flow. This leads to the following system of equations:

$$\mathbf{U}\cdot\nabla\mathbf{U} = -\nabla P + \nabla\cdot[2Re^{-1}\mathbf{D}(\mathbf{U})],$$
$$\nabla\cdot\mathbf{U} = 0,$$

where $Re = \rho U_0 L_0/\mu$ is the Reynolds number. This form assumes that the characteristic velocity of the flow is a convection velocity and implies a balance between pressure gradients and convection terms. This form can easily be implemented in a Navier–Stokes code in primitive variables by setting the density to 1 and the viscosity equal to $1/Re$. This form is inappropriate for creeping flow problems. Indeed, when simulations were performed, the discrete divergence was found to be of the order of 10^{-2} for $Re = 10^{-7}$.

An alternative dimensionless form for creeping flows can be obtained by choosing a viscous diffusion characteristic velocity

$$U_0 = \mu/(\rho L_0),$$

the characteristic length being kept as before. This leads to the set of equations

$$Re(\mathbf{U}\cdot\nabla\mathbf{U}) = -\nabla P + \nabla\cdot[2\mathbf{D}(\mathbf{U})],$$
$$\nabla\cdot\mathbf{U} = \mathbf{0},$$

expressing the proper balance between pressure and viscous forces. This is easily done by setting the density to Re and the viscosity to 1. Use of this form resulted in discrete divergence of the order of machine zero for mixed methods and of order $O(\varepsilon)$ for penalty methods (see Figure 4); results for mixed and penalty methods are identical to within plotting accuracy. Mass flow calculations indicate that for both formulations global mass conservation is exactly satisfied. Thus proper performance of both methods is recovered.

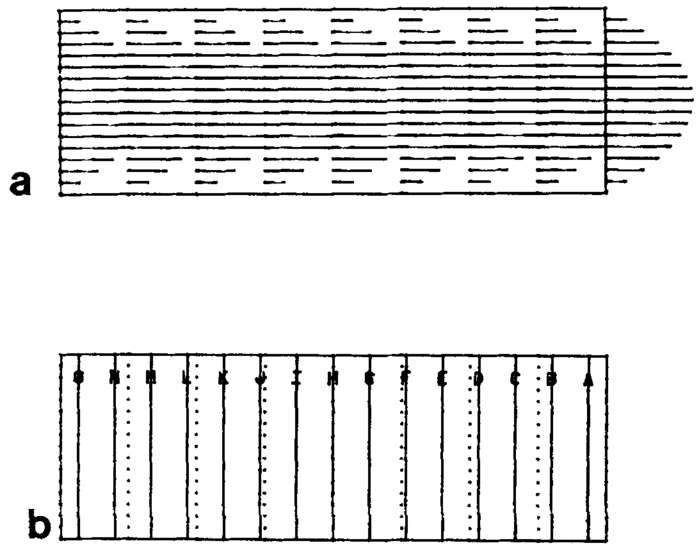

Figure 4. Penalty and mixed solution to proper dimensionless form of the equations: (a) velocity field; (b) pressure contours

The choice of a proper dimensionless form for a given problem is by no means a trivial exercise. For more complex flow problems scale analysis proves to be very useful. The reader is referred to Reference 7 for further details.

Making penalty methods work. Penalty methods are very popular because they result in systems of equations involving only velocity degrees of freedom, thus reducing the computational burden when compared to mixed methods.

It should be remembered that penalty methods enforce the incompressibility constraint in an approximate way only. And when using fluid properties of the previous example, great care should be exercised in selecting the penalty parameter.

Because of the large differences between the terms in (1), a value of the penalty parameter much smaller than is usual must be used. Indeed, setting the penalty parameter to 10^{-12} results in satisfactory velocity and pressure fields. The discrete divergence at the outcome of the simulation was found to be 10^{-7}. A mass flow calculation showed that global mass conservation is satisfied to within 10^{-5}%. This observation was first reported by Hughes *et al.*[8]

There is however a limit to the value of the penalty parameter. Because of the finite precision of the computer, too small a value of the penalty parameter will cause the solution to deteriorate because of ill conditioning and accumulation of round-off errors. Reddy[9] presents an interesting illustration of the effect of the penalty parameter on both the solution and the solution process. The use of such a large penalty term effectively creates a scaling of the viscous and pressure terms in such a way as to make them of equal importance. Note that use of a proper dimensionless form is a safer practice and produces better results than use of stronger penalty constant.

Finally, note that the use of penalty function methods precludes the use of iterative matrix solvers because of the inherently poor conditioning of the resulting system.

The Uzawa algorithm. This technique, an application of the augmented Lagrangian method, is a variant of the penalty method. It contains iterative correction steps that take into account whether the divergence-free condition is satisfied or not and hence ensure satisfaction of the continuity equation. This unduly ignored method is very efficient and powerful and is very popular with the French school of numerical analysis.[10,11]

It is not the purpose of this paper to give a complete account of the method. The interested reader is referred to Reference 10 for a complete description and analysis of the properties of the method. Briefly, the Uzawa algorithm is a gradient method applied to the dual form of the Stokes problem and is given as follows:

1. Choose P_0 as an arbitrary initial pressure (a zero value is very convenient).
2. Let 'n' be the iteration number. Then solve for $n \geqslant 1$ the following problem:

$$\nabla \cdot [2Re^{-1}\mathbf{D}(\mathbf{U}^n)] + \varepsilon^{-1}\nabla(\nabla \cdot \mathbf{U}^n) = -\nabla P^{n-1}.$$

3. Correct the pressure approximation with

$$P^n = P^{n-1} + \varepsilon^{-1}\nabla \cdot \mathbf{U}^n.$$

4. Repeat steps 2 and 3 until the divergence of the velocity field is sufficiently small.

Note that the traditional penalty method corresponds to the case where P_0 is zero and steps 2 and 3 are performed only once. The major advantage of the Uzawa algorithm is that it guarantees satisfaction of the continuity equation to any desired accuracy (machine zero often being used) for any value of the penalty parameter ε. The penalty method will only provide approximate satisfaction of the constraint for very small values of ε.

One must however solve a sequence of Stokes problems (step 2) to obtain the full benefit of the Uzawa method. Fortunately, in most practical cases steps 2 and 3 need only be performed once (as in the traditional penalty method). Note that a pressure gradient term is present on the right-hand side which is absent in the penalty method. However, should the divergence of the flow-field be too large, steps 2 and 3 are repeated until a divergence-free solution is obtained. For very difficult problems little extra cost is incurred compared to the penalty method if the global system of equations is solved by LU decomposition and the factorization is stored.

The implementation of this algorithm is a simple and straightforward modification of any code using the penalty method. Indeed, all the terms appearing in step 2 are already available in any penalty formulation, except for the pressure term which is usually recovered once the velocity is obtained.

The Uzawa algorithm has been generalized to the full Navier–Stokes equations in Reference 12. In this case the technique combines the efficiency of the Uzawa method to satisfy the incompressibility condition and the power of a Newton-based solver to handle the non-linearities. This results in one of the most robust combinations for solving non-linear incompressible flow problems. It provides the accuracy of mixed methods and affords the computational savings of penalty methods.

The strength of the Uzawa method is its ability to produce a divergence-free velocity field even for very poorly conditioned problems such as flows where fluid properties vary by several orders of magnitude. It bypasses the difficult specification of a variable penalty parameter as in the work of Kheshgi and Scriven.[13]

The method was applied to the Poiseuille flow using equations (1) and (2). Although the program had to perform ten Uzawa iterations ($\varepsilon = 10^{-8}$) on this linear problem, it did produce the exact analytical solution. The divergence of the velocity field was of the order 10^{-13}. Mass is conserved to within machine accuracy. This is to be contrasted with the poor solutions obtained with mixed and penalty methods applied to equations (1) and (2). *Note that use of a smaller value of ε significantly reduces the number of iterations.* With $\varepsilon = 10^{-14}$ the same solution was obtained in only two iterations.

Scaling of element matrix equations. Another possibility that was not tested in practice consists in performing the necessary pressure scaling on the element matrix equations. This is especially simple when a discontinuous pressure approximation is used. The weak form of (1) and (2) results in the following system of equations on an element:

$$\begin{bmatrix} \mathbf{K} & -\mathbf{C} \\ -\mathbf{C}^{\mathrm{T}} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{P} \end{bmatrix} = \begin{bmatrix} \mathbf{F} \\ \mathbf{O} \end{bmatrix},$$

where $\mathbf{K}$ is the convection and diffusion matrix, $\mathbf{C}$ is the gradient matrix, $\mathbf{C}^{\mathrm{T}}$ is the divergence matrix, $\mathbf{F}$ is the vector of body forces and boundary conditions, $\mathbf{U}$ is the element velocity vector and $\mathbf{P}$ is the vector of element pressures.

Note in passing that computing the discrete divergence of the velocity field on each element becomes a trivial task if the element matrix equations are stored. One simply has to perform the product $\mathbf{C}^{\mathrm{T}}\mathbf{U}$ to obtain the components of the discrete divergence presented in the previous section.

Difficulties arise when elements of the matrices $\mathbf{K}$ and $\mathbf{C}$ are of greatly different orders of magnitude. One way to perform scaling of the momentum equations consists in multiplying $\mathbf{C}$ and $\mathbf{C}^{\mathrm{T}}$ by a number λ such that $\mathbf{K}$ and $\mathbf{C}$ are of the same magnitude. λ can be chosen by requiring that

$$\|\mathbf{K}\| = \|\lambda \mathbf{C}\|$$

or

$$\lambda = \|\mathbf{K}\| / \|\mathbf{C}\|,$$

where any convenient matrix norm can be used (the max norm for instance).

This is equivalent to a change of variables

$$\mathbf{C}^* = \lambda \mathbf{C}, \tag{5}$$

$$\mathbf{P}^* = \mathbf{P}/\lambda \tag{6}$$

over each element. The value of λ might change from element to element. This results in the following modified system for each element:

$$\begin{bmatrix} \mathbf{K} & -\mathbf{C}^* \\ -\mathbf{C}^{*\mathrm{T}} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{P}^* \end{bmatrix} = \begin{bmatrix} \mathbf{F} \\ \mathbf{0} \end{bmatrix}.$$

The element equations are assembled in the usual fashion and the global system solved by Gaussian elimination.

The original pressure $\mathbf{P}$ is easily recovered in a postprocessing step by solving equation (6), provided that the value of λ has been stored for each element. Note that the scaling resulting from the selection of the appropriate dimensionless form is in fact equivalent to an element equation scaling with a constant value of λ over the whole domain.

THE DISCRETIZATION DILEMMA: MORE COMPLEX FLOWS

We have just seen that round-off errors can have dramatic consequences on the quality of FEM solutions of very simple flows. Some techniques were suggested to avoid some of the pitfalls of incompressible flow simulations. Unfortunately, the use of such good preventive measures will not prevent the occurrence of some difficulties related to the satisfaction of the divergence-free condition. These difficulties are, however, of a more fundamental nature and round-off errors have nothing to do with them. A very careful analysis is then required.

As already pointed out, several authors[1-3] have encountered situations where the finite element solutions were unsatisfactory even for rather simple problems. It is important to note that these difficulties had nothing to do with the celebrated Brezzi–Babuska condition,[10,11,14] since in most cases the finite element approximations used satisfied the condition. In all cases the symptoms were the same, namely a poor satisfaction of the incompressibility condition. Nonsensical velocity distributions were obtained.

To the authors' knowledge, Gresho *et al.*[1] were the first to report such a behaviour for Boussinesq flows. Since then, Debbaut and Crochet[2] and Tidd *et al.* have made similar observations for viscoelastic and free surface problems respectively.

In all cases a continuous pressure approximation was used and the same cure was proposed, namely implementation of discontinuous pressure interpolation. To set the ideas, we briefly recall one no-flow test case introduced by Gresho *et al.*[1] The geometry and boundary conditions are described in Figure 5. The exact solution for this simple problem is given by

$$\mathbf{U}=\mathbf{0}, \qquad P=y^2/2+c, \qquad T=y.$$

In discretizing equations (1)–(3), velocity and temperature were approximated by biquadratic polynomials while a continuous bilinear approximation was used for the pressure. A typical numerical solution is presented in Figure 6 and is obviously not the expected physical one. Such solutions led several authors[2,3] to claim that the root of the problem lies with the fact that continuous pressure approximations make it impossible to satisfy mass conservation at the element level.

Recall that if V_h and Q_h denote the approximation spaces for velocity and pressure, the discrete divergence-free condition is (for $\mathbf{U}_h \in \mathbf{V}_h$)

$$\int_\Omega (q_h \nabla \cdot \mathbf{U}_h)\,\mathrm{d}x = 0 \quad \text{for every } q_h \in Q_h. \tag{7}$$

Note that if one chooses for Q_h a space of piecewise continuous polynomials (i.e. pressure can be discontinuous across element boundaries), we have in particular (if Q_h contains piecewise constant pressure)

$$\int_E (\nabla \cdot \mathbf{U}_h)\,\mathrm{d}x = 0, \tag{8}$$

where E denotes an element in the domain. Equation (8) is a statement of element mass conservation. It is unfortunately not the case for continuous approximations. Indeed, the closest

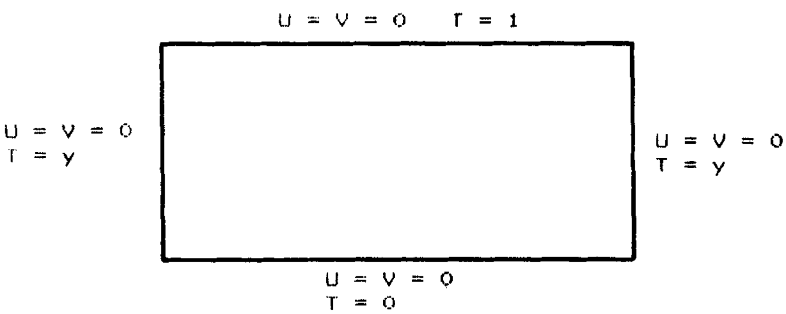

Figure 5. Problem statement for the no-flow case of Reference 1

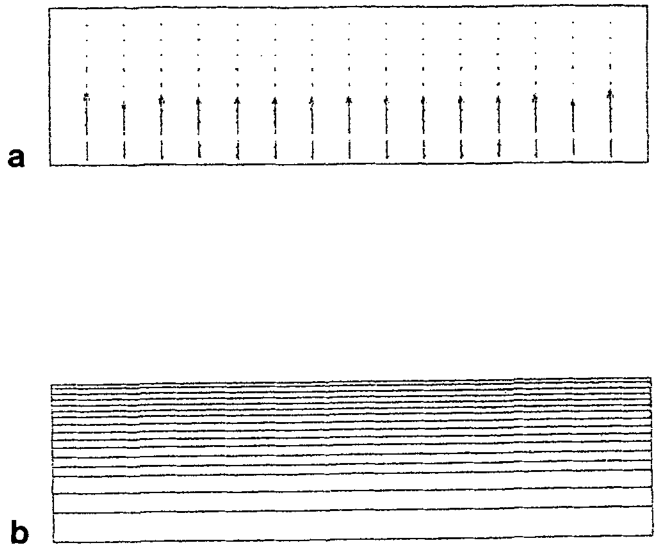

Figure 6. The no-flow test case of Reference 1, Q2/Q1 solution: (a) velocity field; (b) pressure contours

approximation to equation (8) one can achieve is

$$\int_{\Omega_i} (q_i \nabla \cdot \mathbf{U}_h)\, \mathrm{d}x = 0,$$

where Ω_i is the set of all elements adjacent to the ith pressure node. Some authors[2,3] claimed that violation of equation (8) was the cause of the poor results observed. Gresho *et al.*[1] pointed out that this is only partly true. To gain deeper insight, let us take a closer look at the no-flow test. Equations (1) and (2) simplify to

$$\nabla P = \mathbf{S}, \qquad \nabla \cdot \mathbf{U} = 0$$

or equivalently in variational form

$$-(P, \nabla \cdot \mathbf{v}_h) = (\mathbf{S}, \mathbf{v}_h) \qquad \text{for all } \mathbf{v}_h \in V_h,$$
$$(q_h, \nabla \cdot \mathbf{U}_h) = 0 \qquad \text{for all } q_h \in Q_h.$$

It is now clear that the pressure plays a dual role: it balances the source term due to buoyancy and enforces the incompressibility condition. Hence there must be enough pressure degrees of freedom in order to achieve this numerically. In other words, the pressure space Q_h must be sufficiently rich. Consequently, modifying the Q2/Q1 element by adding a pressure degree of freedom defined on each element as done in Reference 1 (thus making the pressure discontinuous across element boundaries) is simply a very efficient and elegant way to enrich the pressure space. Another way consists in using an appropriate discretization, such as the Q2/P1, from the very beginning. Figure 7 shows the dramatic improvement resulting from the use of the Q2/P1 element.

A consequence of passing from a continuous to a discontinuous pressure approximation is that mass is conserved over each element (equation (8) is satisfied). On the other hand, using a higher-order pressure approximation consistent with the Brezzi condition would also improve the results. Moreover, from a mathematical point of view, an upper bound for the error of the finite element

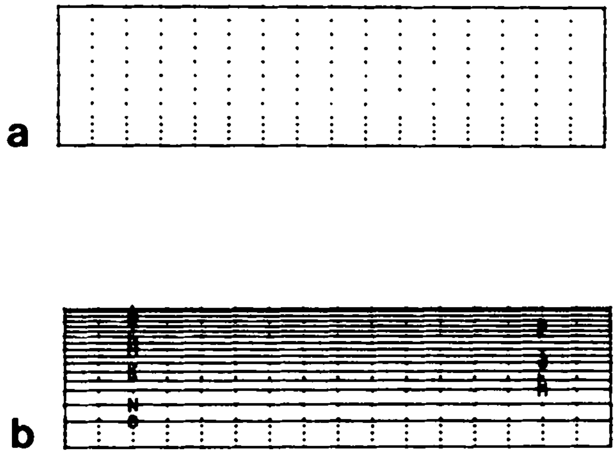

Figure 7. The no-flow test case of Reference 1, Q2/P1 solution: (a) velocity field; (b) pressure contours

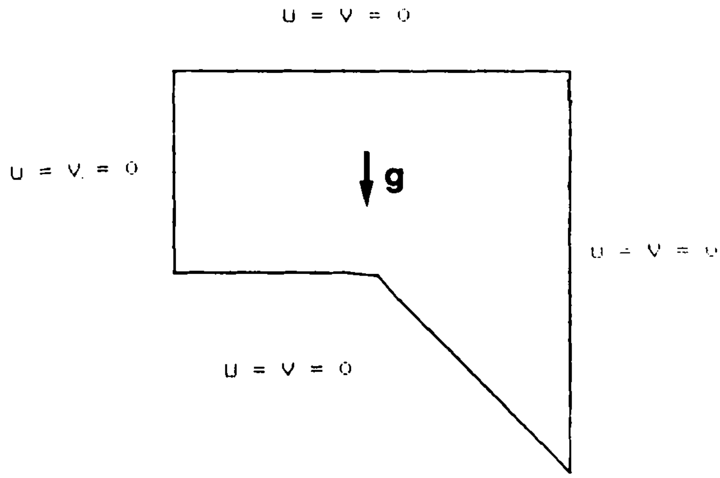

Figure 8. Problem statement for the no-flow test case of References 14 and 15

solution is given by[11]

$$\|\mathbf{U}-\mathbf{U}_h\| \leqslant \inf_{\mathbf{v}_h \in \mathbf{V}_h} \|\mathbf{U}-\mathbf{v}_h\| + \inf_{q_h \in Q_h} |P-q_h|, \tag{9}$$

where $\|\cdot\|$ is the norm on the space $\mathbf{V}_h$ and $|\cdot|$ is the norm on the space Q_h.

Equation (9) states that the quality of the numerical solution is controlled by the quality of both the velocity and pressure approximations. If there are not enough pressure degrees of freedom, the error in the pressure (the second term on the right-hand side of equation (9)) can be large, thus causing, according to equation (9), a large error in the numerical velocity solution. In other words, *a poor pressure approximation can cause important velocity errors.* Note that equation (9) is always true and has nothing to do with whether or not the finite element scheme guarantees mass conservation at the element level. Another simple example will clarify this statement.

Consider the Stokes flow problem described in Figure 8. This problem was first considered by

Gresho *et al.*[15] and studied in detail by Fortin and Fortin.[14] For this problem the source term is the gravitational force and is given by

$$\mathbf{S} = (0, g).$$

The exact solution is given by

$$\mathbf{U} = \mathbf{0}, \qquad P = gy + b.$$

If the problem is solved with the bilinear velocity/constant pressure element of Q1/P0 (which ensures mass conservation on the element), a nonsensical solution is obtained (see Figure 9). This is a clear example that element mass conservation is not a sufficient condition to guarantee accurate velocity solutions.

Note that if a finite element scheme with linear pressure is used, the exact solution is obtained. Indeed, the pressure approximation space is rich enough to exactly balance the source term. Thus no error is committed on the pressure and equation (9) states that the velocity error will be zero to machine accuracy. Figure 10 illustrates this last point.

CONCLUSIONS

We have shown on a simple problem that failure to satisfy mass conservation can occur with mixed and penalty methods. *The calculation of the discrete divergence of the velocity field is a very reliable measure of the incompressibility of the solution.* Several techniques have been presented to alleviate the problem: choosing an appropriate dimensionless form, using a high enough penalty parameter with penalty methods, implementing an Uzawa algorithm to iteratively satisfy the constraint, and performing a scaling of the element matrix equations.

The combination of element matrix scaling and the Uzawa algorithm is probably the most robust combination. It guarantees satisfaction of the constraint to machine accuracy. For those computer programs implementing the more traditional mixed and penalty methods, *inclusion of divergence computation is such a straightforward and inexpensive modification that it should not be overlooked.* This diagnostic tool can save the analyst many aggravations.

For more complex flows the pressure often plays a dual role: it enforces incompressibility and balances some other effects. If the pressure approximation is not rich enough, nonsensical velocity

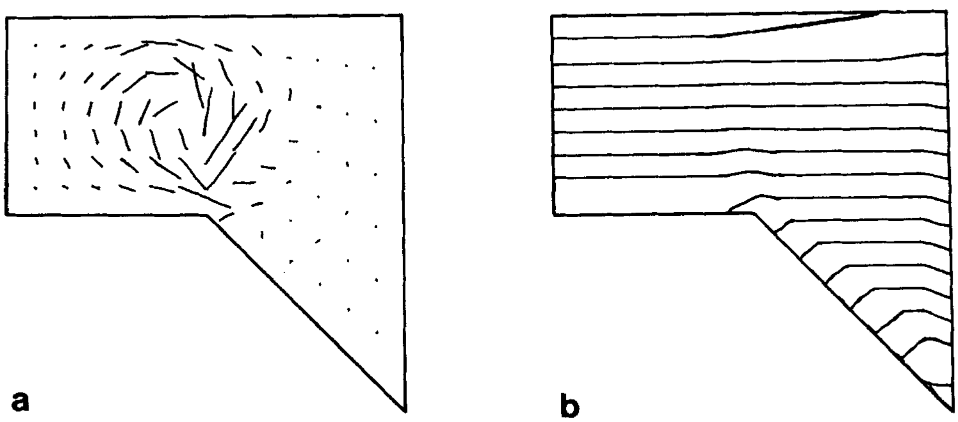

Figure 9. The no-flow test case of References 14 and 15, Q1/P0 solution: (a) velocity field; (b) pressure contours

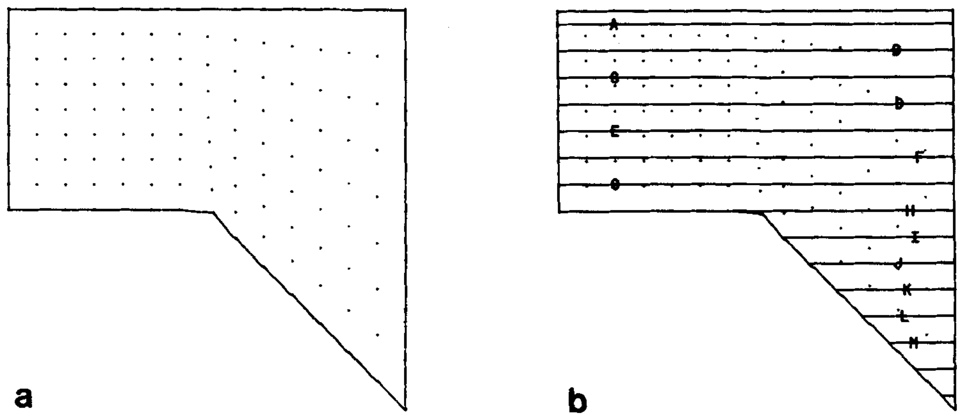

Figure 10. The no-flow test case of References 14 and 15, Q2/P1 solution: (a) velocity field; (b) pressure contours

fields can occur. Thus, *even when one is mostly interested in the velocity distribution for a specific problem, an accurate pressure approximation is still of prime importance.* Various experiences[1-3] have clearly demonstrated that continuous pressure interpolations are unable to accommodate the dual role of the pressure. Discontinuous pressure approximations yield much richer spaces Q_h and, at the same time, better velocity fields. This is especially true if the velocity and pressure interpolations satisfy the Brezzi compatibility condition. *Hence we recommend the use of discontinuous pressures, particularly for strongly coupled flow problems.* It is then possible to implement the techniques described previously to minimize the effect of round-off errors. Moreover, discontinuous pressures permit the use of the computationally efficient penalty or Uzawa methods.

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